

# **Simulating Fluids Using Fast Diagonalization**

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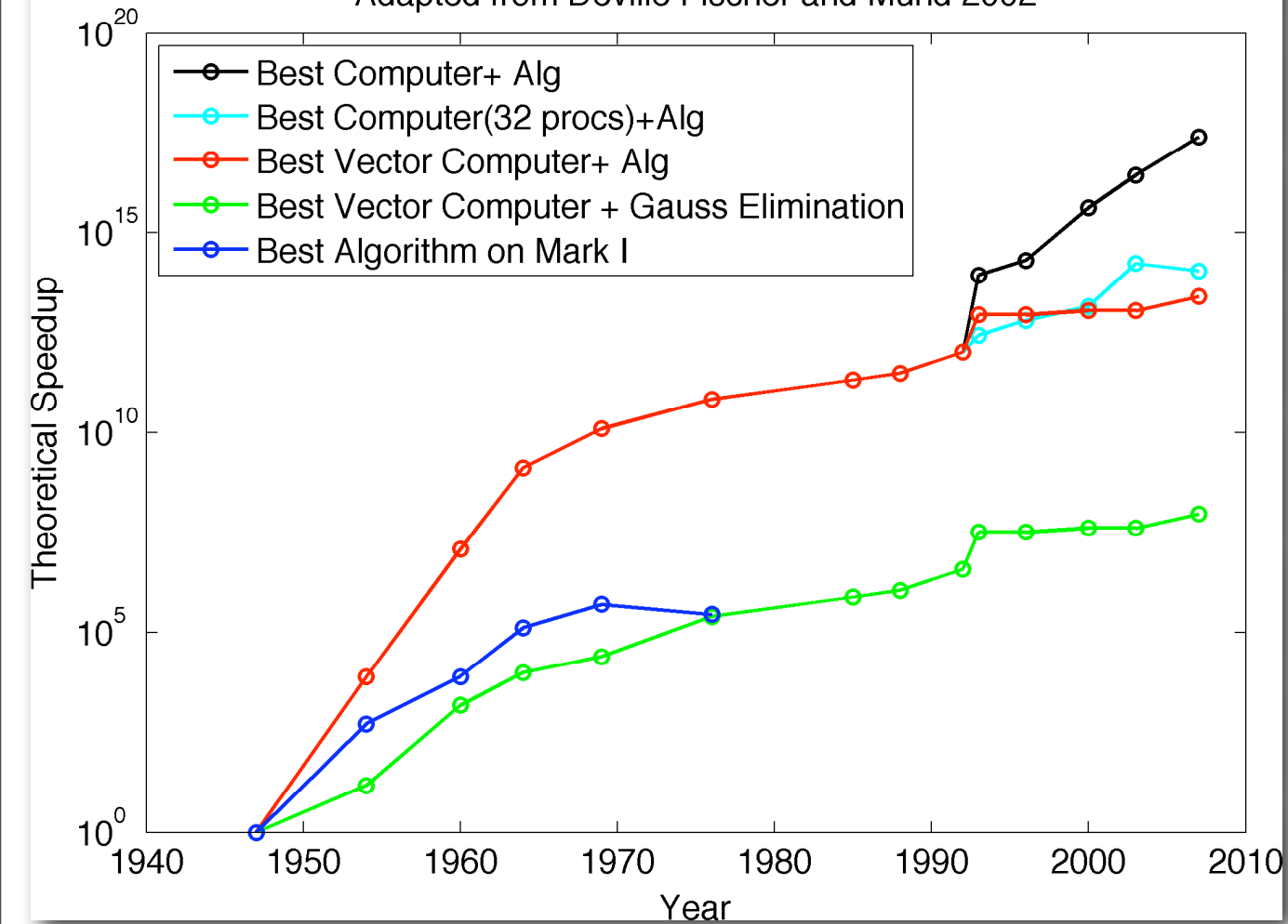
**Applied Mathematics & Scientific Computation  
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**Joint work with:**

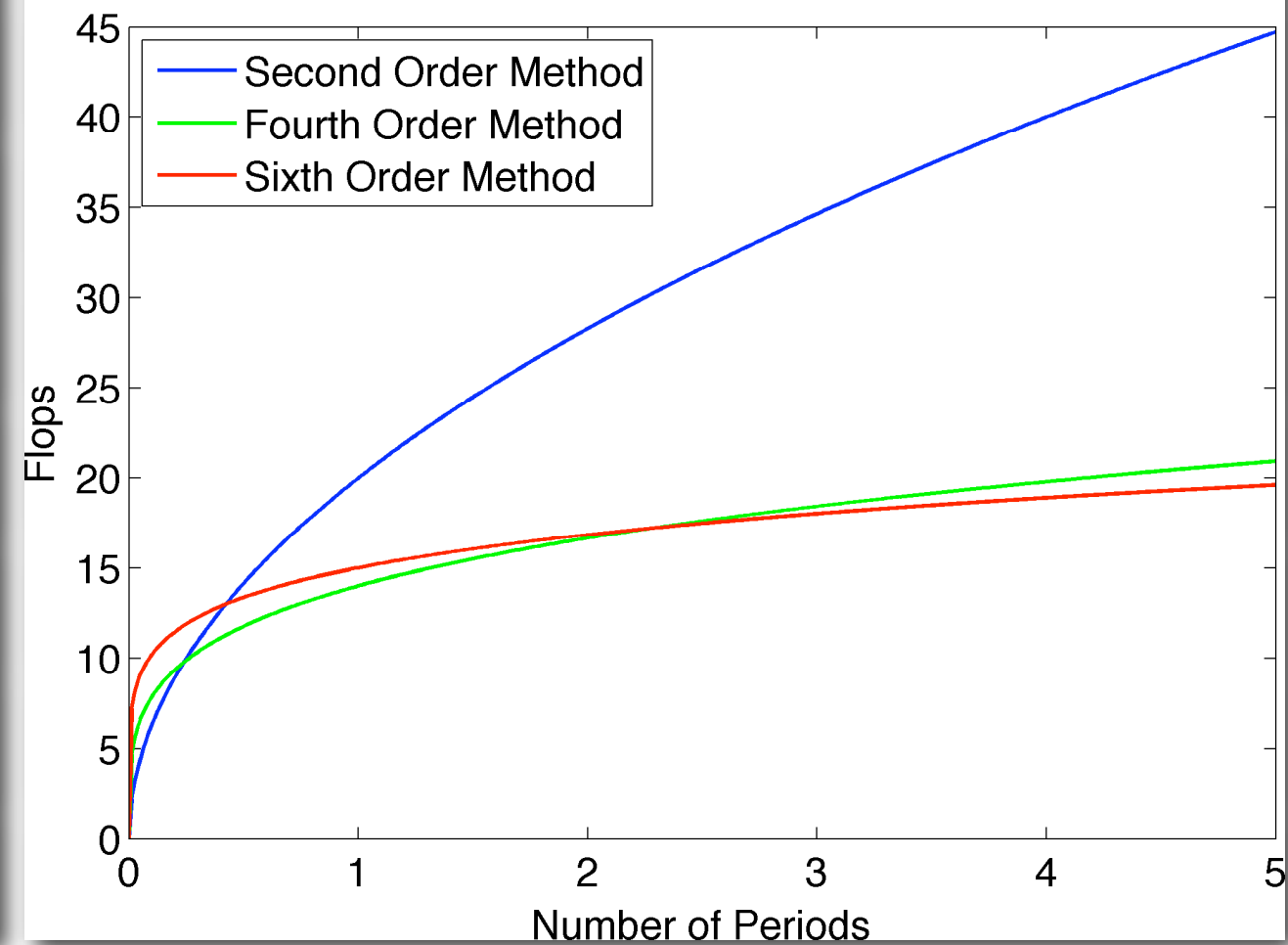
**Howard Elman (CS) & Anil Deane (IPST)**

# Motivation - Efficient Solvers & Discretization

Evolution of machines, algorithms and their combination over the past 60 years for the solution of a 3D Poisson Eqn  
Adapted from Deville Fischer and Mund 2002



Comparison of computational work needed to maintain 10% phase error in 1D advection equation  
Karniadakis & Sherwin 2005



Faster machines and computational algorithms can dramatically reduce simulation time.

High order based discretizations can be used to obtain accurate, efficient simulations.

# Model - Steady Advection Diffusion

$$-\epsilon \nabla^2 u + (\vec{w} \cdot \nabla) u = f$$

Inertial and viscous forces occur on disparate scales causing **sharp flow features** which:

- require fine numerical grid resolution
- cause poorly conditioned systems.

These properties make solving the discrete systems computationally expensive.

# Methods - Spectral Element Discretization

A spectral element discretization provides:

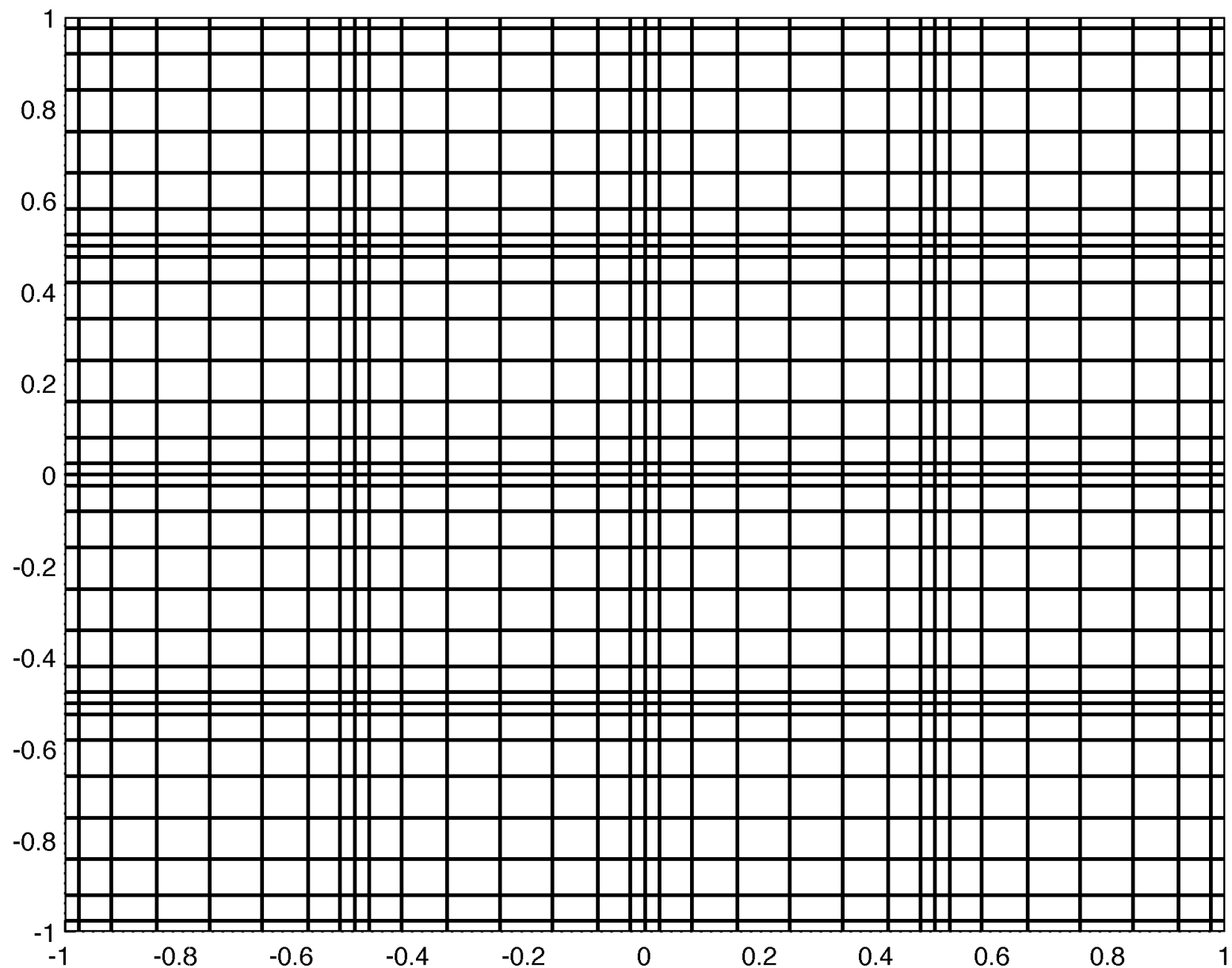
- **accurate element based discretization**
- **large volume to surface ratio**

$$F(\vec{w})u = Mf$$

For bi-constant winds, we can use:

- **fast diagonalization**
- **minimal memory**

$$\tilde{F} = \hat{M} \otimes \hat{F}(w_x) + \hat{F}(w_y) \otimes \hat{M}$$



# Methods -Tensor Products

What does  $\otimes$  mean?

Suppose  $A_{k \times l}$  and  $B_{m \times n}$   
then the Kronecker Tensor Product

$$C_{km \times ln} = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1l}B \\ a_{21}B & a_{22}B & \dots & a_{2l}B \\ \vdots & \vdots & & \vdots \\ a_{k1}B & a_{k2}B & \dots & a_{kl}B \end{pmatrix}.$$

Matrices of this form have some great properties that make computations **very efficient** and **save lots of memory**!

## Methods - Fast Diagonalization

Matrix-vector multiplies can be recast as  $(A \otimes B)\vec{u} = BU A^T$

which can be done in  $O(n^3)$  operations instead of  $O(n^4)$  (even better savings in 3D)

Best of all though is the Fast Diagonalization Property

$$C = A \otimes B + B \otimes A$$

Under certain conditions on A & B we can diagonalize C easily

$$V^T A V = \Lambda, \quad V^T B V = I$$

$$C = (V \otimes V)(I \otimes \Lambda + \Lambda \otimes I)(V^T \otimes V^T)$$

$$C^{-1} = (V \otimes V)(I \otimes \Lambda + \Lambda \otimes I)^{-1}(V^T \otimes V^T)$$

Only need an inverse of a **diagonal matrix!**

# Methods - Solver & Preconditioner

The discrete system of equations is solved iteratively using Flexible GMRES.

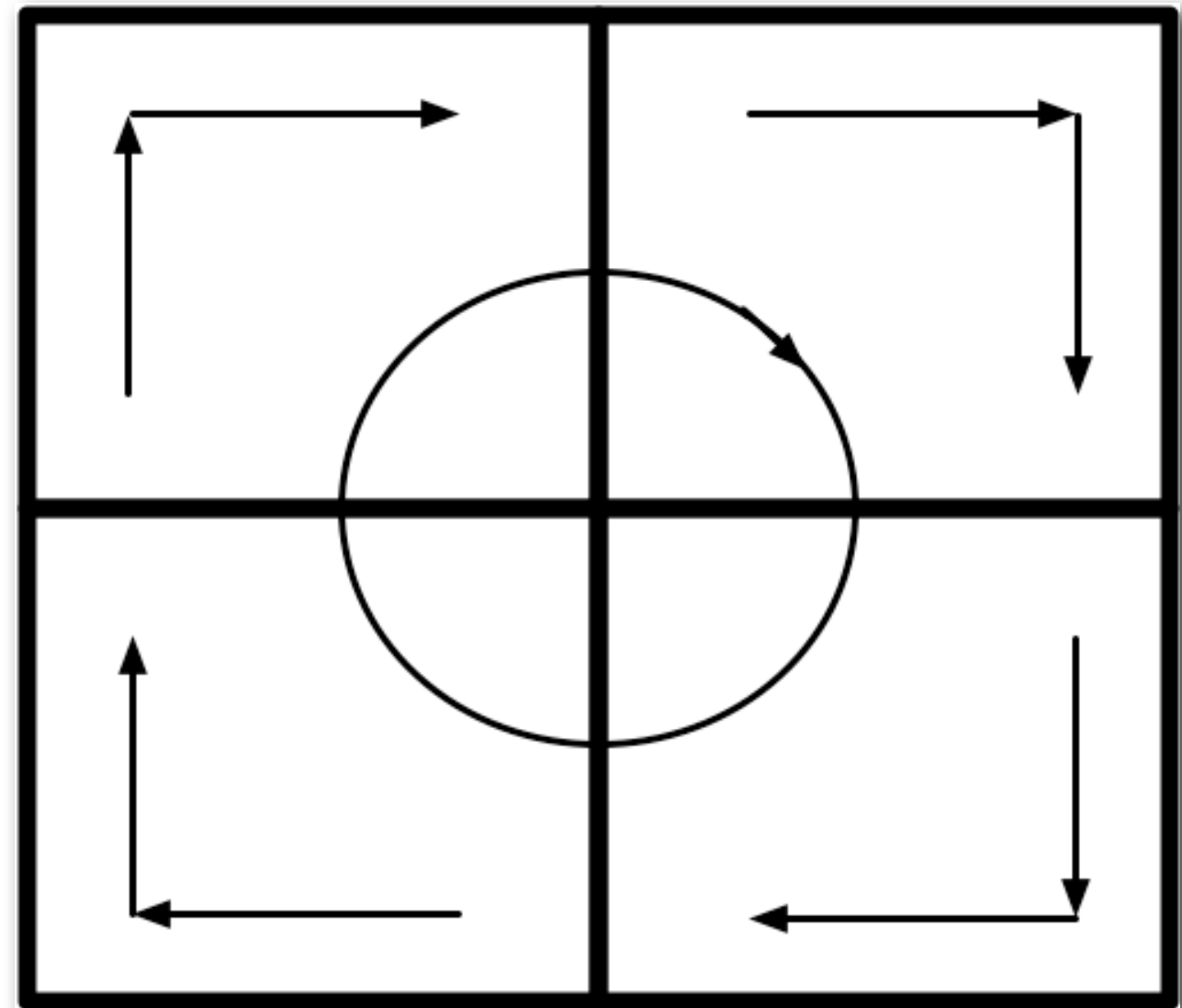
We construct a **preconditioner** based on:

- **Bi-constant wind approximations**
- **Fast Diagonalization**
- **Domain Decomposition**

$$F(\vec{w})P_F^{-1}P_F u = Mf$$

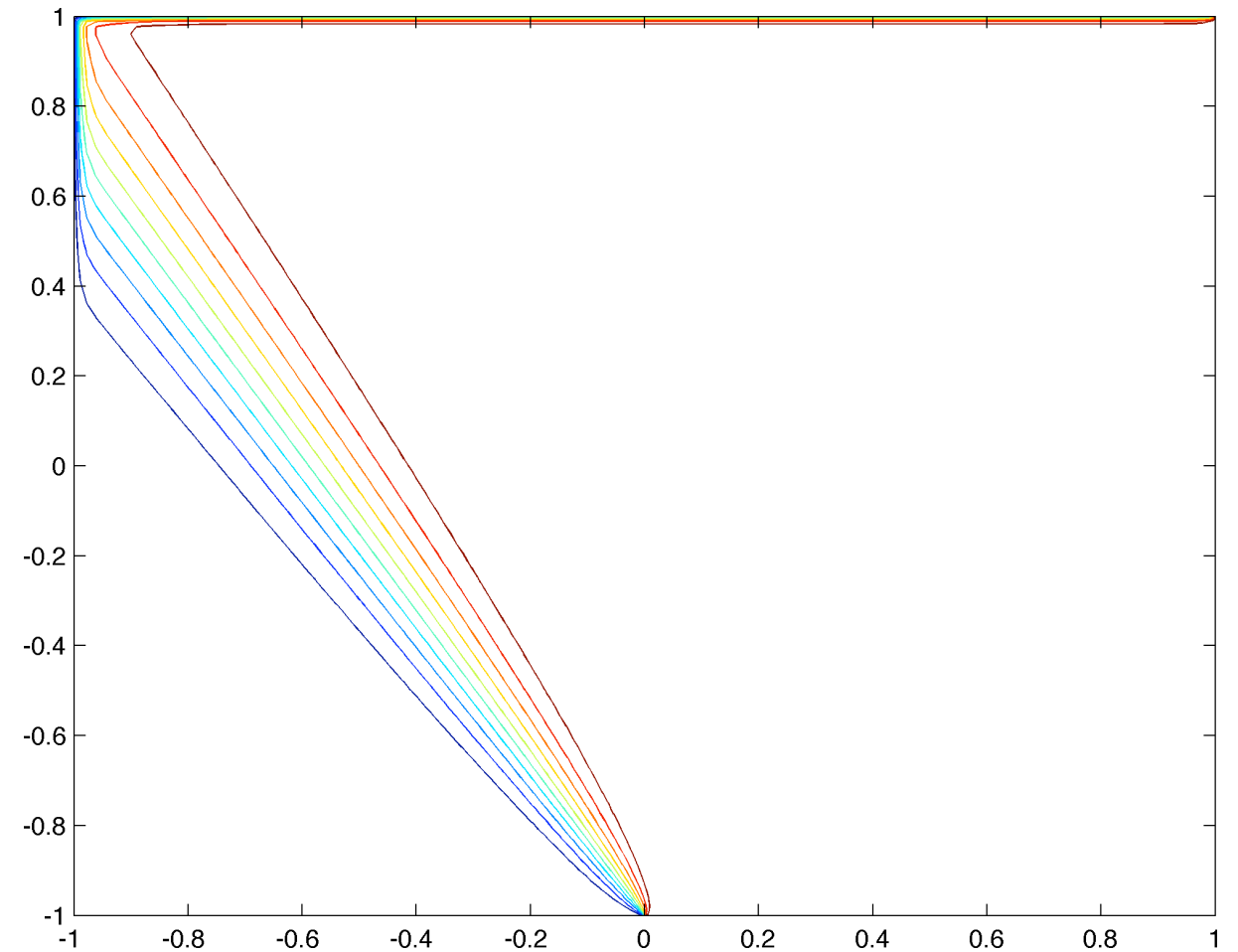
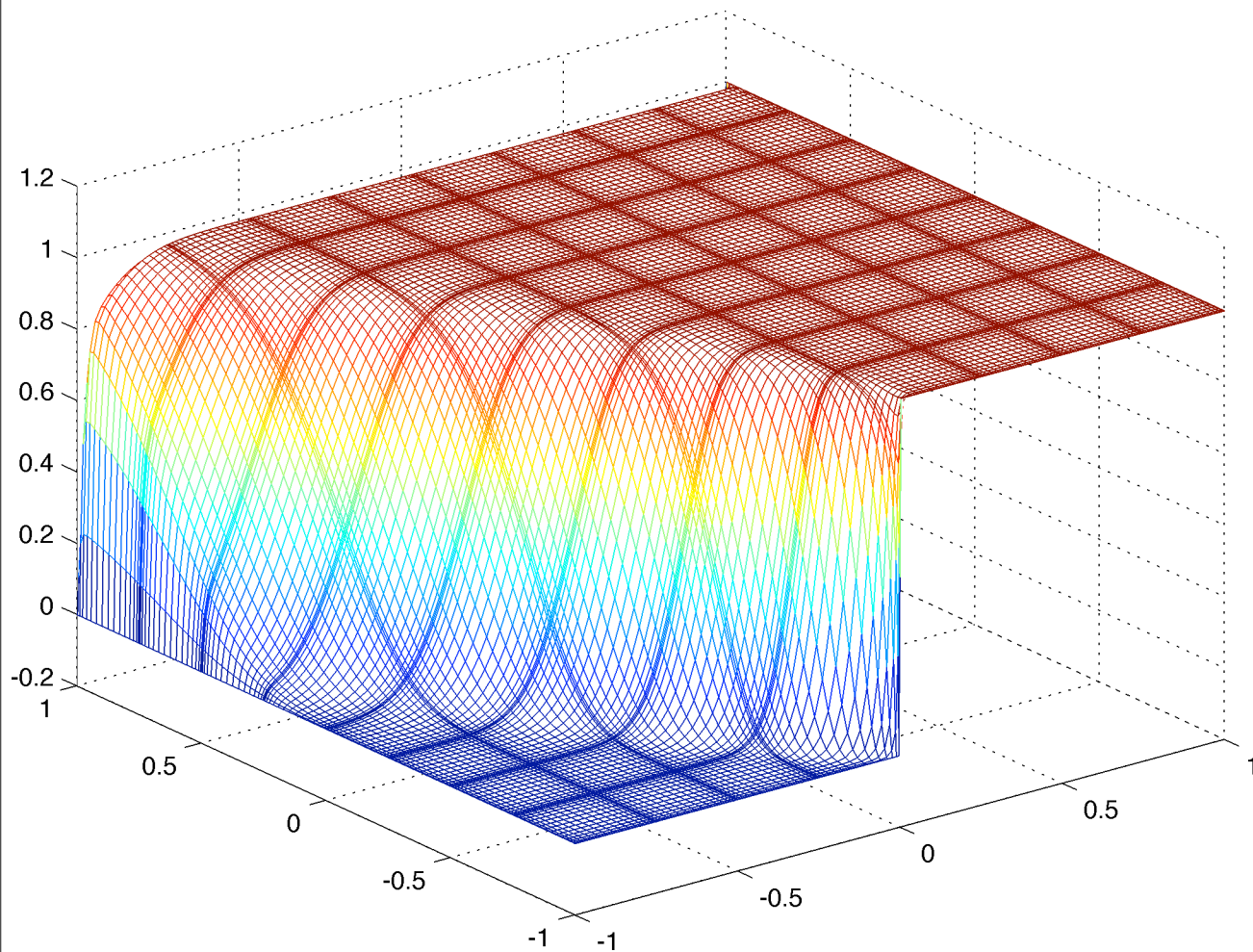
$$P_F^{-1} = R_0^T \tilde{F}_0^{-1}(\bar{w}_0)R_0 + \sum_{e=1}^N R_e^T \tilde{F}_e^{-1}(\bar{w}^e)R_e$$

$$\tilde{F}_e^{-1} = (\hat{M}^{-1/2} \otimes \hat{M}^{-1/2})(S \otimes T)(\Lambda \otimes I + I \otimes V)^{-1}(S^{-1} \otimes T^{-1})(\hat{M}^{-1/2} \otimes \hat{M}^{-1/2})$$



# Solver Results - Bi-constant Wind

$$\vec{w} = 200\left(-\sin\left(\frac{\pi}{6}\right), \cos\left(\frac{\pi}{6}\right)\right)$$

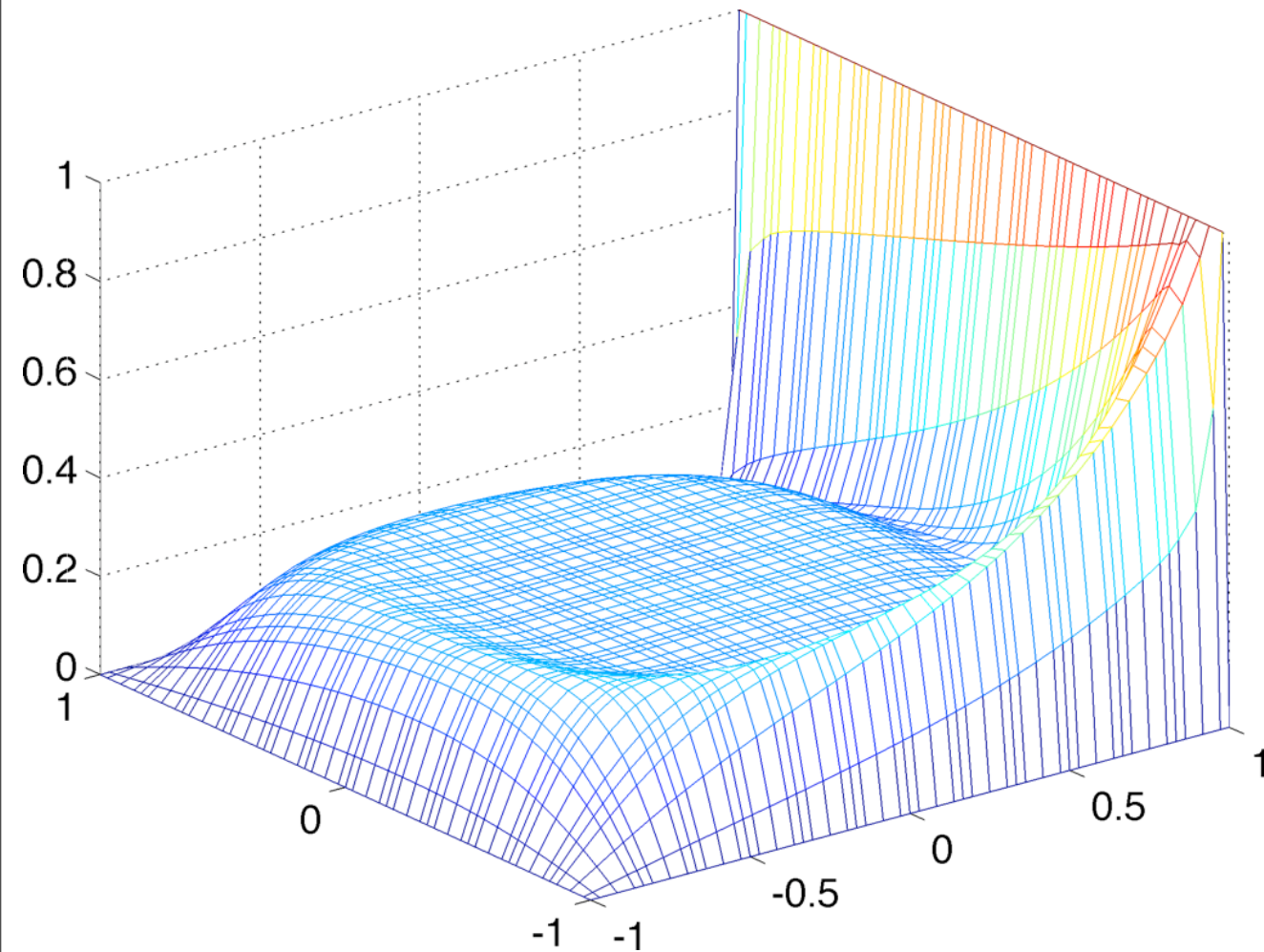


Solution and contour plots of a steady advection-diffusion flow with bi-constant wind using Domain Decomposition & Fast Diagonalization as an exact solver. Interface solve takes 150 steps to obtain  $10^{-5}$  accuracy.



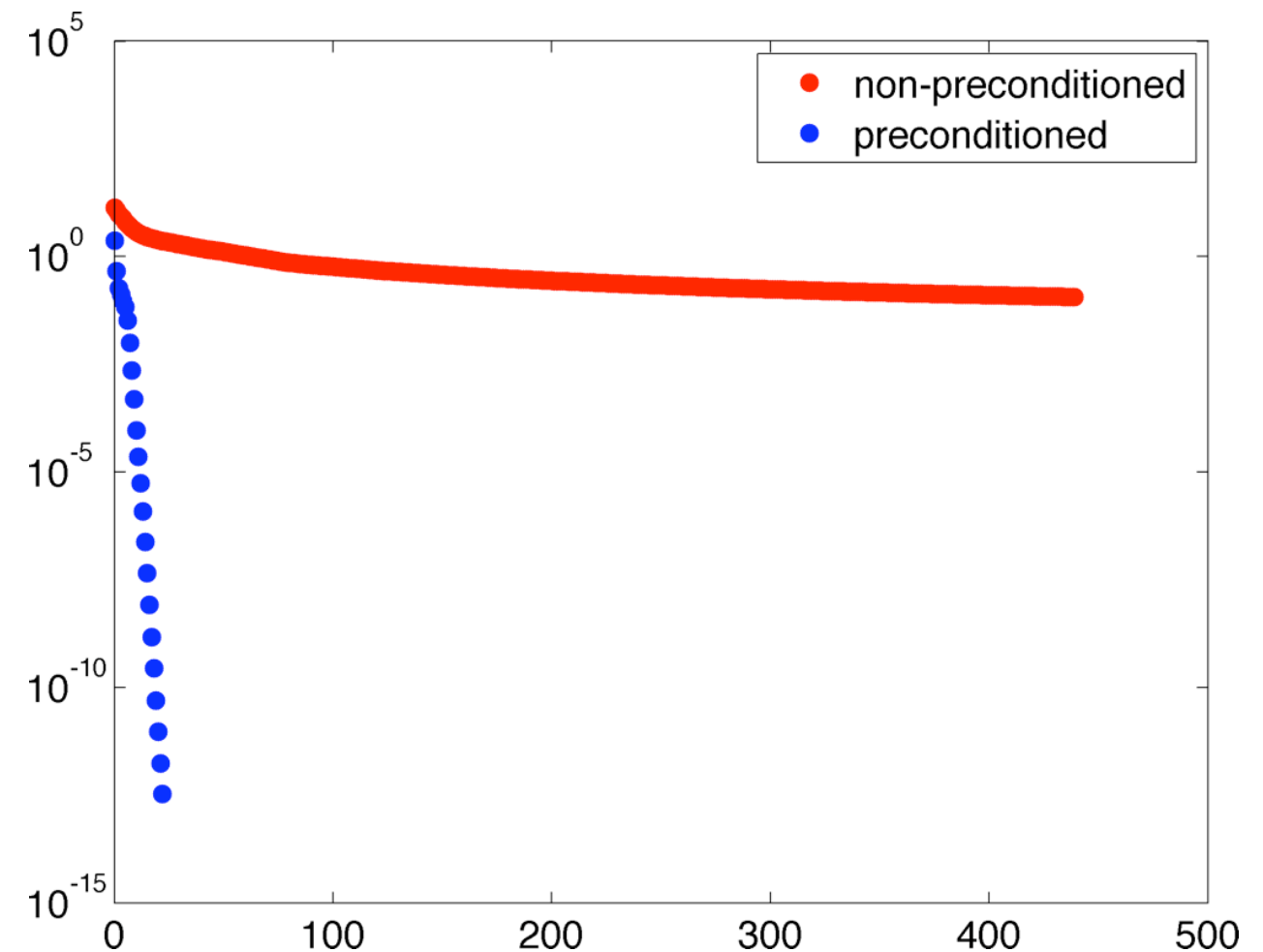
# Preconditioner Results - Recirculating Wind

$$\vec{w} = 200(y(1 - x^2), -x(1 - y^2))$$



Steady advection-diffusion flow with recirculating wind.

Hot plate at wall results in sharp internal boundary layer.



Comparison of iteration residuals.

- 30 interface steps yield 10% accuracy
- $(P+1)[120N+(P+1)]$  additional flops per step

# Future Directions

- Precondition Interface Solve
- Analytical FDM
- Multiple wind directions per element
- 2D & 3D **Navier-Stokes**
- Flows with boundary layers

# References

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- H. Elman, D. Silvester, & A. Wathen, Finite Elements and Fast Iterative Solvers with applications in incompressible fluid dynamics, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2005.
- H. Elman, P.A. Lott Matrix-free preconditioner for the steady advection-diffusion equation with spectral element discretization. In preparation. 2008.
- H. Elman, P.A. Lott Matrix-free Block preconditioner for the steady Navier-Stokes equations with spectral element discretization. In preparation. 2008.